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Logocyclic Curve by a continuous motion; and a very ingenious instrument has been contrived by Mr. Henry Johnson of Crutched Friars, to describe the spiral of Archimedes, which is as simple as it is effective.

*June* 17, 1858.

The LORD WROTTESLEY, President, in the Chair.

The Earl Granville, Professor Hennessy, and the Rev. Samuel Haughton were admitted into the Society.

In accordance with notice given at the last Meeting, the Earl of Rosse proposed the Right Hon. Sir John Pakington, Bart. for election and immediate ballot.

The Ballot having been taken, Sir John Pakington was declared duly elected.

The following communications were read:—

I. "On the Problem of Three Bodies." By CHARLES JAMES HARGREAVE, LL.D., F.R.S. Received May 3, 1858.

(Abstract.)

The author states that the principal object of this memoir is to set forth two new methods of treating the dynamical equations by processes of variation of elements, differing from the ordinary processes of this nature principally in this particular, that the variations are represented in explicit terms of the elements themselves and of the time, and not through the medium of partial differential coefficients. It has been his object to render the processes as elementary as possible; and to preserve them in a vigorous form, by postponing all attempts at approximation until the formulæ are actually applied to practical problems. The applications given in the paper comprise the circular and spherical pendulums, and the planetary and lunar theories, and a special theorem as to the movement of the plane of a planet's motion under the influence of several other planets.

The original normal problem which is taken as the basis, is that

of motion about a fixed centre of force, where the force is directly as the distance; or, in other words, the system of equations not exceeding three in number, of the form

$$x''+n^2x=0;$$

whose solutions are represented under the form

$$\begin{aligned}
x&=\lambda_a\, a\cos(nt+\rho)+\mu_a\, b\sin(nt+\rho),\\
y&=\lambda_b\, a\cos(nt+\rho)+\mu_b\, b\sin(nt+\rho),\\
z&=\lambda_c\, a\cos(nt+\rho)+\mu_c\, b\sin(nt+\rho);
\end{aligned}$$

where

$$\begin{aligned}
\lambda_a&=\cos\phi\cos\psi-\sin\phi\sin\psi\cos l,\\
\lambda_b&=\cos\phi\sin\psi+\sin\phi\cos\psi\cos l,\\
\lambda_c&=\sin\phi\sin l;
\end{aligned}$$

$$\begin{aligned}
\mu_a&=-\sin\phi\cos\psi-\cos\phi\sin\psi\cos l,\\
\mu_b&=-\sin\phi\sin\psi+\cos\phi\cos\psi\cos l,\\
\mu_c&=\cos\phi\sin l;
\end{aligned}$$

to which are afterwards added,

$$\begin{aligned}
\nu_a&=\sin\psi\sin l,\\
\nu_b&=-\cos\psi\sin l,\\
\nu_c&=\cos l.
\end{aligned}$$

These are the equations of an ellipse whose centre is at the force, and situated in a plane inclined at the angle $l$ to the plane of $x\,y$, and the longitude of whose node is $\psi$; and $\phi$ is the angular distance of the major axis of the ellipse from the node; $a$ and $b$ are the semi-axes of the ellipse; and $\rho$ is the angular distance, from the major axis, of the zero-point of the motion, measured on the circle described on the major axis. A uniform motion around the circle represents the place of the body by the corresponding point on the ellipse, where it is cut by a perpendicular dropped on the major axis.

If the force be not situated at the origin, but at the point (X, Y, Z), we have merely to substitute $x-\mathrm{X}$ for $x$, &c. in the above equations of motion and solutions.

Applying the method of tangential variation to the system

$$x''+\frac{\mu}{r^3}x=0,\ \&c.,$$

we perceive that this system admits of complete solution in finite terms, leading in fact to the usual theory of elliptical motion. Taking this system, therefore, as a normal system, the author proceeds to

deduce the formulæ for the variation of the elements of this system, in order to arrive at the solution of the system

$$x'' + \frac{\mu}{r^3} x = \mathrm{P}_x, \text{ \&c.}$$

The elements which have been selected, for reasons fully explained in the paper, are $l$ and $\psi$, whose meanings are already known; A and $\mathrm{N}\tau$ denoting respectively the mean distance, and the longitude of the epoch measured in the plane of the tangential ellipse as it exists at the time $t$, and measured from the node at that time; and $e$ and $\varpi$ denoting respectively the eccentricity of the tangential ellipse, and the longitude of its perihelion measured as above; and it is observed that these are strictly normal elements, according to Professor Donkin's definition of normal elements.

The variations of these elements are then rigorously found, and are expressed as follows:—Denote

$$\cos\psi\,\mathrm{P}_x + \sin\psi\,\mathrm{P}_y \text{ by the symbol } \mathrm{P}_\xi,$$

and

$$\cos l\,(\cos\psi\mathrm{P}_x - \sin\psi\mathrm{P}_y) + \mathrm{P}_z \sin l \text{ by the symbol } \mathrm{P}_\eta;$$

and let

$$-\mathrm{P}_\xi \sin\theta + \mathrm{P}_\eta \cos\theta = \mathrm{P}_{\xi,\theta}; \quad -\mathrm{P}_\xi \sin\varpi + \mathrm{P}_\eta \cos\varpi = \mathrm{P}_{\xi,\omega};$$
$$\mathrm{P}_\xi \cos\theta + \mathrm{P}_\eta \sin\theta = \mathrm{P}_{\eta,\theta}; \quad \mathrm{P}_\xi \cos\varpi + \mathrm{P}_\eta \sin\varpi = \mathrm{P}_{\eta,\omega};$$

then

$$\delta l = \frac{r\cos\theta}{\mathrm{NA}^2(1-e^2)^{\frac{1}{2}}}(\mathrm{P}_z),$$

$$\sin l\,\delta\psi = \frac{r\sin\theta}{\mathrm{NA}^2(1-e^2)^{\frac{1}{2}}}(\mathrm{P}_z),$$

$$\delta e = \frac{r}{\mathrm{NA}^2(1-e^2)^{\frac{1}{2}}}\left\{(\cos(\theta-\varpi)+e)\,\mathrm{P}_{\xi,\theta} + (1+e\cos(\theta-\varpi))\,\mathrm{P}_{\xi,\omega}\right\},$$

$$e(\delta\varpi + \cos l\,\delta\psi) = \frac{r}{\mathrm{NA}^2(1-e^2)^{\frac{1}{2}}}\left\{\sin(\theta-\varpi)\,\mathrm{P}_{\xi,\theta} - (1+e\cos(\theta-\varpi))\,\mathrm{P}_{\eta,\omega}\right\},$$

$$\delta\mathrm{A} = \frac{2}{\mathrm{N}(1-e^2)^{\frac{1}{2}}}(\mathrm{P}_{\xi,\theta} + e\,\mathrm{P}_{\xi,\omega}),$$

$$\delta(\mathrm{N}(t+\tau)) = \frac{r}{\mathrm{NA}^2 e}\left\{-\sin(\theta-\varpi)\,\mathrm{P}_{\xi,\theta} + (1+e\cos(\theta-\varpi))\,\mathrm{P}_{\eta,\omega} - 2e\,\mathrm{P}_{\eta,\theta}\right\};$$

which are capable of being expanded in terms of the elements, and $t$ by means of the ordinary expressions for $r$, $\theta$, and $\theta-\varpi$ in terms of the same quantities. The values of the elements at the time $t$ being supposed to be found, by the integration of these formulæ, in terms

of $t$, and their initial values, are to be substituted in the ordinary expressions for the coordinates, so as to obtain their values at the time $t$.

The author exhibits the application of the preceding formulæ to certain simple examples, and then proceeds to apply them to the planetary theory. For two planets (distinguished by the suffixes 2 and 3) supposed to move in the same plane, the following are the rigorous expressions for the variations. Let $\alpha_2$ and $\alpha_3$ be the ratio of the mass of each planet to that of the central body. Let P denote the cube of $r_3 \div r_{23}$, and let $(P-1)\sin(\theta_3-\theta_2)$ be called Q, and $(P-1)\cos(\theta_3-\theta_2)-\frac{r_2}{r_3}P$ be called R; then

$$\delta e_2 = \alpha_3 \frac{N_2 A_2 r_2}{r_3^2 (1-e_2^2)^{\frac{1}{2}}} \left\{ (\cos(\theta_2-\varpi_2)\,(2+e_2\cos(\theta_2-\varpi_2))+e_2)\,Q + \sin(\theta_2-\varpi_2)\,(1+e_2\cos(\theta_2-\varpi_2))R \right\},$$

$$e_2\,\delta\varpi_2 = \alpha_3 \frac{N_2 A_2 r_2}{r_3^2 (1-e_2^2)^{\frac{1}{2}}} \left\{ \sin(\theta_2-\varpi_2)\,(2+e_2\cos(\theta_2-\varpi_2))\,Q - \cos(\theta_2-\varpi_2)\,(1+e_2\cos(\theta_2-\varpi_2))\,R \right\},$$

$$\delta A_2 = 2\alpha_3 \frac{N_2 A_2^3}{r_3^2 (1-e_2^2)^{\frac{1}{2}}} \left\{ (1+e_2\cos(\theta_2-\varpi_2))\,Q + e_2\sin(\theta_2-\varpi_2)\,R \right\},$$

$$\delta(N(t+\tau)) = -(1-e^2)^{\frac{1}{2}}\,\delta\varpi - 2\alpha_3 \frac{N_2 A_2 r_2}{r_3^2} R.$$

From these formulæ, the secular variations of the elements are obtained without difficulty; and a new method of integrating the equations for the variations of the eccentricity and longitude of perihelion is given.

The author then enters upon a minute examination of the mathematical character of secular variations, and their bearing upon the methods of approximation to which the problem of three bodies has given rise. It is pointed out that the disturbance finally effected through the medium of a secular variation is not of the order of the disturbing force, or rather of the ratio of the disturbing force to the central force; but that it may remain precisely the same, though this ratio should be diminished or increased without limit. The difference affects not the aggregate amount of deviation or disturbance caused, but the time in which this aggregate amount is produced. If we consider the undisturbed problem of two planets about a sun

as representing motion in two planes inclined to each other at the angle I, and in ellipses having eccentricities $e_2$ and $e_3$, it is shown that, no matter how small or how large may be the disturbing force produced on each orbit by the other planet, the aggregate amount of disturbance of the planet $m_2$ is of the order of the quantities I and $e_2$, and that of the planet $m_3$, of the order of I and $e_3$. From considerations of this nature, which are dwelt upon at length in the memoir, the author concludes that the ordinary direct methods of solution by approximation, being based upon the erroneous assumption that the variations of the coordinates are of the order of the disturbing force, are not, in a mathematical sense, legitimate processes; and that, in the planetary theory, they produce results practically true only on account of the minuteness of the disturbing forces, and the consequent great length of the secular periods; and that, in the lunar theory, their failure is made evident, in consequence of the comparatively large magnitude of the disturbing force, and the consequent rapidity with which the elements of the moon's orbit pass through their secular periods.

The formulæ for the variations of the elements are then applied to the lunar theory; and some of the integrations are effected by means of a lemma containing the solution of the differential equation

$$\frac{d\phi}{dt}=\mathrm{F}\cos(pt-q\phi)$$

(where F, $p$ and $q$ are numerical coefficients), in the form

$$\cos(pt-q\phi)=\frac{q\mathrm{F}+p\cos \mathrm{M}t}{p+q\mathrm{F}\cos \mathrm{M}t},\ \mathrm{M}\text{ being }\left(1-\frac{q^2\mathrm{F}^2}{p^2}\right)^{\frac{1}{2}}p.$$

By this method, the total motion of the moon's perigee, as well as the coefficients of the evection, are fully obtained in the first instance, without the necessity of any second approximation; and the usual difficulty as to the movement of the perigee does not present itself. The motion of the node, and the evection in latitude, are correctly obtained in a similar manner.

This part of the memoir is concluded by an extension of the general formulæ for the tangential variation of elements to the case in which we suppose the constant $\mu$ to become variable, the result being to add to each variation a term involving $\delta\mu$.

The third part of the Paper contains the development of the

method of osculating variation, before briefly described; from which are deduced the formulæ for the osculating variations of elliptic elements. This method is capable of being applied to the planetary and lunar theories, as well as that of tangential variation; but the advantages of this method did not appear to be such as to justify the actual expansion of the formulæ for these theories. The author, however, shows that with reference to any system of three bodies, the equations of motion for each body naturally assume the form

$$\chi'' + n^2(x - \mathrm{X}) = 0, \text{ \&c.}$$

(being the system solved by this method); and that the X, Y, and Z are absolutely the same for each of the three bodies. This is shown by demonstrating, that at any given moment the three lines which represent the direction of the force acting on each of the three bodies all pass through the same point, which is denominated the centre of force. The coordinates of this common centre of force are,

$$\mathrm{X} = \frac{(23)x, + (31)x_2 + (12)x_3}{(23) + (31) + 12};$$

with similar expressions for Y and Z; (12) being $r_{12} \div m\, m_2 \chi r_{12}$, $\chi$ denoting the law of force, &c. Each body has its own value of $n^2$; their ratios being denoted by the proportion

$$n_1^2 : n_2^2 : n_3^2 : : \frac{r_{23}}{\chi r_{23}} : \frac{r_{31}}{\chi r_{31}} : \frac{r_{12}}{\chi r_{12}}.$$

The invariable plane of this system of three bodies is then found; and it is shown that the nodes of the three orbits upon this plane are always in a certain relative position, constituting a kind of triangle of equilibrium about the centre of force; resulting, in the limiting case where one of the three bodies is infinitely larger than the other two (or in what is denominated the undisturbed Problem of Three Bodies), in an exact opposition of the two nodes of the orbits of the latter two bodies upon the invariable plane of the system.

The formulæ for the osculating variation of elements are then applied to a system of three bodies, of which one possesses a predominating magnitude, so far as is necessary to determine the movement of the planes of the orbits; and it is readily shown that, if we consider only the first order of the disturbing force, the inclination

of the plane of each orbit to the invariable plane is absolutely constant; and that the two nodes are always in opposition to each other, and move with a uniform angular velocity round the invariable plane.

This theorem is then extended to a system of $n$ bodies moving about a central predominant body; and it is shown that the aggregate effect of the disturbing forces of such a system upon the plane of any one of the bodies can always be represented by stating that its node upon a certain fixed plane revolves with a uniform angular velocity, the plane of the orbit always remaining at the same inclination to the fixed plane. The rate of this angular movement, and the coordinates of the fixed plane upon which the movement takes place, are found by means of formulæ of remarkable simplicity. These three quantities may be ascertained once for all for each planet (viz. the inclination of the fixed plane on which the node moves to any coordinate plane, the longitude of the node of the fixed plane in relation to any coordinate line, and the angular rate of movement of the node of the orbit upon this fixed plane), and, when once ascertained, may be regarded as fixed elements of the planet, from which the position of the plane of its orbit can always be determined without the use of tables.

It is then shown that a system of the form

$$x''+n^2x=\mathrm{P}_x, \text{ \&c.},$$

where $n^2$ and $\mathrm{P}_x$, $\mathrm{P}_y$, and $\mathrm{P}_z$ are any variables, may be solved by the same set of final integrals, and the same values of $x'$, $y'$, and $z'$, by supposing the elements $a$, $b$, $\phi$, $\psi$, $z$, $l$, and $\rho$ to become variable. These elements are those of an ellipse tangential to the actual curve of motion; and the following formulæ are obtained for their variation:—

Let

$$\begin{aligned}
\lambda_a \mathrm{P}_x+\lambda_b \mathrm{P}_y+\lambda_c \mathrm{P}_z&=(\mathrm{P}_x),\\
\mu_a \mathrm{P}_x+\mu_b \mathrm{P}_y+\mu_c \mathrm{P}_z&=(\mathrm{P}_y),\\
\nu_a \mathrm{P}_x+\nu_b \mathrm{P}_y+\nu_c \mathrm{P}_z&=(\mathrm{P}_z);
\end{aligned}$$

and let (putting T for $nt+\rho$)

$$\begin{aligned}
a\cos\phi\cos\mathrm{T}-b\sin\phi\sin\mathrm{T}&=\xi,\\
a\sin\phi\cos\mathrm{T}+b\cos\phi\sin\mathrm{T}&=\eta;
\end{aligned}$$

then

$$nab\,\delta l=\xi(\mathrm{P}_z),$$

$$nab\sin l\,\delta\psi=\eta(\mathrm{P}_z),$$

$$\delta(nab)=a\cos\mathrm{T}(\mathrm{P}_y)-b\sin\mathrm{T}(\mathrm{P}_x),$$

$$\tfrac{1}{2}\delta n^2(a^2+b^2)=-n(a\sin\mathrm{T}(\mathrm{P}_x)-b\cos\mathrm{T}(\mathrm{P}_y))+r^2nn',$$

$$(a^2-b^2)\,(\delta\phi+\cos l\,\delta\psi)=\frac{1}{n}(b\cos\mathrm{T}(\mathrm{P}_x)-a\sin\mathrm{T}(\mathrm{P}_y))+2a\,b\sin\mathrm{T}\cos\mathrm{T}\frac{n'}{n},$$

$$(a^2-b^2)\,\delta(nt+\rho)=-\frac{1}{n}(a\cos\mathrm{T}(\mathrm{P}_x)-b\sin\mathrm{T}(\mathrm{P}_y))-(a^2+b^2)\sin\mathrm{T}\cos\mathrm{T}\frac{n'}{n}.$$

It may be observed that $\xi$ and $\eta$ are coordinates of the body referred to the plane of the tangential ellipse, and to an axis of $\xi$ coinciding with the node.

This method is denominated the method of Tangential Variation; and it is applied directly to the problem of the circular pendulum, that of the spherical pendulum, and that of the motion of a particle where the force is a function of the distance, and in particular that of elliptical motion, where the law of force is that of the inverse square.

In a subsequent part of the paper it is shown that a system of the form

$$x''+n^2(x-\mathrm{X})=0,\ \&c.,$$

where $n^2$, X, Y, and Z are any variables, may be solved by the same set of final integrals, and the same values of $x'$, $y'$, and $z'$ as those which have been already given as the solutions of the same system when $n$, X, Y, and Z are constant, by supposing the elements to become variable. In such a case, the elements are those of an ellipse osculating with the actual curve of motion, always of course having its centre at the moveable point (X Y Z). The following formulæ are obtained for the variation of these elements:—

Let

$$\lambda_a\mathrm{X}'+\lambda_b\mathrm{Y}'+\lambda_c\mathrm{Z}'=(\mathrm{X}'),$$

$$\mu_a\mathrm{X}'+\mu_b\mathrm{Y}'+\mu_c\mathrm{Z}'=(\mathrm{Y}'),$$

$$\nu_a\mathrm{X}'+\nu_b\mathrm{Y}'+\nu_c\mathrm{Z}'=(\mathrm{Z}');$$

then

$$nab\,\delta l=(\mathrm{Z}')\frac{d\xi}{dt},$$

$$nab\sin l\,\delta\psi=(\mathrm{Z}')\frac{d\eta}{dt},$$

$$\delta(nab) = -n((X')\, b \cos T + (Y')\, a \sin T),$$

$$\tfrac{1}{2}\delta(n^2(a^2+b^2)) = -n^2((X')\, a \cos T + (Y')\, b \sin T) + r^2 n n',$$

$$(a^2-b^2)\,(\delta\phi + \cos l\, \delta\psi) = -((X')\, b \sin T + (Y')\, a \cos T) + 2a\, b \sin T \cos T \frac{n'}{n},$$

$$(a^2-b^2)\, \delta(nt+\rho) = (X')\, a \sin T + (Y')\, b \cos T - (a^2+b^2) \sin T \cos T \frac{n'}{n};$$

in which $\frac{d\xi}{dt}$ and $\frac{d\eta}{dt}$ are the differential coefficients of the expressions for $\xi$ and $\eta$, taken explicitly with regard to $t$.

This method is denominated the method of Osculating Variation.

II. "Description of some Remains of a Gigantic Land-Lizard (*Megalania prisca*, Ow.) from Australia." By Prof. Richard Owen, F.R.S. Received May 13, 1858.

(Abstract.)

The subject of this communication forms part of a collection of fossil remains from Australia, recently acquired by the British Museum, and demonstrates the former existence in that continent of a land-lizard considerably surpassing in bulk the largest species now known. The characters are chiefly derived from vertebræ, partially fossilized, equalling in size those of the largest existing Crocodiles; they are of the 'procœlian' type, but present lacertian modifications, and closely agree with those in the great existing 'Lace-lizard' of Australia (*Hydrosaurus giganteus*, Gray), of which individuals upwards of six feet long have been taken. A generic or subgeneric distinction is indicated by the comparatively contracted area of the neural canal, and by the inferior development of the neural spine, of the fossil vertebræ, which have belonged to an individual not less than twenty feet in length, calculated from the vertebræ and proportions of the body of the existing *Hydrosauri*. For this, probably extinct lizard, the name of *Megalania prisca* is proposed.

The results of an extended series of comparisons of its vertebræ with those of recent and extinct Sauria are given; and the paper is illustrated by drawings of the vertebræ of *Megalania* and those of *Hydrosaurus*.